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Geometric Structures Approximated by Maxwell's Equations

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We show that classical electrodynamics can be obtained as a limit of a system of "geodesic" equations on 2-vector fields in an Artinian manifold. The limit method is geometrically analogous to the method used to obtain Newtonian mechanics as the limit of the geodesic equations on a Lorentzian manifold. It is also shown that the current and energy-momentum conservation law of electrodynamics can be obtained directly from the "geodesic" formulation. The geometric structures introduced are related to semi-Kählerian and balanced structures in complex geometry.

INTRODUCTION

It is a well-known and physically important fact that Newton's second law approximates the geodesic equations in the limit of small velocities. It is this fact that makes metric geometry a physically viable extension of classical mechanics. This article seeks to demonstrate that there is a formally analogous construction in electrodynamics whereby Maxwellian electrodynamics is also realized as the limit of a more invariant geometric structure.

When Newtonian mechanics is given a geometric formulation, it is found to be a metric geometry together with a totally geodesic codimension-one foliation determined by a function called the time function that defines the universal time of each leaf of the foliation. The gradient of the time function is a Killing vector field for the metric geometry. Although the existence of such a function is a strong condition on a metric geometry, in any metric geometry it is possible to find along a given trajectory a function that reduces the geodesic equation instantaneously to Newton's second law. To see that an analogous reduction is possible in electrodynamics, one must first identify the auxiliary structure comparable to the time function of Newtonian mechanics. From a mechanical point of view (Souriau, 1970),

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it is well known that the carrier of the 2-form that represents the electromagnetic field is not the space-time manifold, but is rather the cotangent bundle of the space-time manifold. However, to distinguish those 2-forms on the cotangent bundle that correspond to electromagnetic fields requires an auxiliary structure, namely the vertical polarization and the canonical symplectic structure. It will be shown that this pair of structures is the analog of the codimension-one foliation and time function of Newtonian mechanics. In fact, on an Artinian manifold there is a system of partial differential equations that reduces to Maxwell's equations in the presence of a local symplectic structure with a local polarization.

The presentation of these results is organized as follows: The first section introduces a class of dynamical structures that is the basis for both the electromagnetic and mechanical models studied in this article. These systems can be described as extensions of the geodesic equations that arise from the Carathéodory approach to the calculus of variations. In the second section the reduction of the geodesic equation in Fermi normal coordinates to the Newtonian force law is reexamined using the dynamical formalism introduced in Section 1. Although this material is not new, it is included because it provides the setting for the reduction of a similar geodesic system of equations to Maxwell's equations presented in Section 3. In Section 4 is is shown that this geodesic system possesses analogs of the current and energy-momentum differential conservation laws.

1. PRELIMINARY RESULTS AND NOTATIONS

Because many of the following calculations depend explicitly on the choice of normalization of the exterior calculus, it may be helpful to recall the standard normalization convention. The r-fold exterior power of a real vector space V is defined by $\Lambda^r(V) = T^r(V)/\mathscr{A}$, where $T^r(V)$ is the r-fold tensor product of V and $\mathscr{A} = \ker(\text{Alt})$. Here Alt is the projection on $T^r(V)$ defined as the alternating sum of the isomorphisms t_σ of $T^r(V)$ induced by a permutation on r letters $\sigma \in \mathscr{P}_r$; that is, for $w \in T^r(V)$,

$$\text{Alt}(w) = \frac{1}{r!} \sum_{\sigma \in \mathscr{P}_r} \text{sign}(\sigma) t_\sigma(w) \tag{1.1}$$

The elements of $\Lambda^r(V)$ are called r-vectors and the elements of $\Lambda^r(V^*)$ are called r-forms. Recall that the dual space $T^r(V)^*$ can be identified with $T^r(V^*)$ using the pairing between $\lambda_1 \otimes \cdots \otimes \lambda_r \in T^r(V^*)$ and $v_1 \otimes \cdots \otimes v_r \in T^r(V)$ given by $\lambda_1 \otimes \cdots \otimes \lambda_r(v_1 \otimes \cdots \otimes v_r) = \lambda_1(v_1) \ldots \lambda_r(v_r)$. This pairing can also be viewed as a multilinear map denoted by $\lambda_1 \otimes \cdots \otimes \lambda_r(v_1, \ldots, v_r) = \lambda_1 \otimes \cdots \otimes \lambda_r(v_1 \otimes \cdots \otimes v_r)$. Using this representation of the dual space of $T^r(V)$, we can also identify the dual space $\Lambda^r(V)^*$ with $\Lambda^r(V^*)$.

Recall that the exterior algebra of V is the algebra defined on $\Lambda(V) = \bigoplus_{r=0} \Lambda^r(V)$ by the wedge product. The wedge product between $\Sigma \in \Lambda^r(V)$ and $M \in \Lambda^s(V)$ is given by

$$\Sigma \wedge M = \frac{(r+s)}{r!\,s!} \operatorname{Alt}(\Sigma \otimes M)$$

The notion of a multilinear map can be extended to arbitrary exterior powers of V; that is, for $\lambda \in \Lambda^r(V^*)$ and $\Sigma_j \in \Lambda^{i_j}(V)$ with $j \in (1,\ldots, k)$ and $i_1 + \cdots + i_k = r$ define $\lambda(\Sigma_1,\ldots, \Sigma_k) = \lambda(\Sigma_1 \otimes \cdots \otimes \Sigma_k)$. From (1.1) it follows that

$$\lambda(\Sigma_1 \wedge \cdots \wedge \Sigma_k) = \frac{r!}{i_1! \ldots i_k!} \lambda(\Sigma_1, \ldots, \Sigma_k)$$

Also define the interior product between $\Sigma \in \Lambda^k(V)$ and $\lambda \in \Lambda^r(V^*)$ as follows. When $r \geq k$ define $\iota(\Sigma)\lambda \in \Lambda^{r-k}(V^*)$ by $\iota(\Sigma)\lambda(M) = \lambda(\Sigma, M)$ for $M \in \Lambda^{r-k}(V)$, and if $k > r$, then $\iota(\lambda)\Sigma \in \Lambda^{k-r}(V)$ is given by

$$\binom{k}{r} \mu(\iota(\lambda)\Sigma) = \lambda \wedge \mu(\Sigma) \qquad \text{for} \quad \mu \in \Lambda^{k-r}(V^*)$$

A k-vector Σ and a k-form λ determine an endomorphism $\mathscr{C}(\Sigma, \lambda)$ of V. The concomitant $\mathscr{C}(\Sigma, \lambda)$ is obtained by contracting the first $k-1$ indices of Σ with the first $k-1$ indices of λ. On simple vectors it is given by the expression

$$\begin{aligned}&\mathscr{C}(v_1 \wedge \cdots \wedge v_k, \lambda_1 \wedge \cdots \wedge \lambda_k)\\ &\quad= \sum_{i,j} (-1)^{i+j} \lambda_1 \wedge \cdots \wedge \hat{\lambda}_i \wedge \cdots \wedge \lambda_k (v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k) v_j \otimes \lambda_i\end{aligned}$$

The concomitant $\mathscr{C}$ is an essential object in the following constructions. It arises partly because it satisfies the following identity for the interior product. Let $\Sigma \in \Lambda^k(V)$ and let $\lambda \in \Lambda^1(V^*)$ and $\sigma \in \Lambda^k(V^*)$; then

$$\iota(\Sigma)\sigma \wedge \lambda = \sigma(\Sigma)\lambda - k\lambda(\mathscr{C}(\Sigma, \sigma)) \tag{1.2}$$

Finally, recall that a k-vector Σ or a k-form σ is said to be nondegenerate if for any $\lambda \in V^*$ of $v \in V$, $\iota(\lambda)\Sigma = 0$ implies $\lambda = 0$ or $\iota(v)\sigma = 0$ implies $v = 0$. Nondegeneracy is a generic condition $\Lambda^k(V)$ or $\Lambda^k(V^*)$ for $k > 2$. In the case of 2-vectors or 2-forms nondegeneracy is generic if $\dim(V)$ is even.

If M is a smooth manifold, then the previous considerations apply to the tensor products of the tangent bundle of M. Denote the bundle of k-vectors over M by $\Lambda_k(M)$ and the bundle of k-forms over M by $\Lambda^k(M)$. The space of sections of $\Lambda_k(M)$ or the space of k-vector fields on M is denoted by $\mathscr{E}_k(M)$, with the exception that the space of vector fields will be denoted by $\mathscr{X}(M)$. The space of sections of $\Lambda^k(M)$ or the space of differential k-forms is denoted by $\mathscr{E}^k(M)$. This article studies a class of dynamical structures on 2-vector fields. These structures were introduced for k-vector fields in Martin (1988). To make this article somewhat self-contained, I will recall the definitions of Martin (1988) and briefly describe the results needed for the present discussion.

Dynamical structures on k-vector fields naturally arise in the Carathéodory approach to the first-order variational calculus. In this construction the variational problem is formulated in terms of an immersion $f: N \to M$, where M and N are m-dimensional and n-dimensional smooth manifolds and N is compact with boundary. The differential of an immersion f is encoded in the image by f of fixed n-vector field Γ on N. Since $f_*\Gamma$ is a section of $\Lambda^n(M)$ along N, the Lagrangian function L is then a degree-one homogeneous function on $\Lambda_n(M)$. If σ is the unique n-form on N such that $\sigma(\Gamma) = 1$, then the action integral determined by L has the form

$$\mathscr{A}(f) = \int_N L(f_*\Gamma)\sigma$$

The best illustration of a variational problem that is naturally formulated from Carathéodory's point of view is the minimal surface problem. In this case L is the function on $\Lambda_n(M)$ determined by the length function on the fiber of $\Lambda_n(M)$ induced by the metric on M.

Just as in mechanics, the Euler–Lagrange equations for f can be given an invariant formulation in terms of the natural geometry of $\Lambda_n(M)$. On $\Lambda_n(M)$ there is a canonical nondegenerate differential $(n+1)$-form γ defined by the relation $\gamma = d\alpha$, where α is the tautological form on $\Lambda_n(M)$. Recall that the value of the tautological n-form α at $p \in \Lambda_n(M)$ on $v_1, \ldots, v_n \in T(\Lambda_n(M))_p$ is $\alpha_p(v_1, \ldots, v_n) = p(\pi_* v_1, \ldots, \pi_* v_n)$.

Next, with the Lagrangian function L associate the Legendre map $l: \Lambda_n(M) \to \Lambda^n(M)$. To define l, let d_v denote the vertical derivative on functions along the fibers $\Lambda_n(M)$ and let, for $u \in \Lambda_n(M)$, i^{-1}: $\Lambda_n(M)_p \to T(\Lambda_n(M)_p)_u$ be the natural identification. In terms of these operations $l(u)$ evaluated on $w \in \Lambda_n(M)_p$ is given by $l(u)(w) = d_v L_u(i^{-1}w)$. As in mechanics, there exists an n-vector field Σ on $\Lambda^n(M)$ satisfying the conditions that $l(\pi_*\Sigma_\lambda) = \lambda$ for $\lambda \in l(\Lambda_n(M))$ and $d\iota(\Sigma)\gamma = 0$. The n-vector field Σ is called the Hamiltonian n-vector field for the Lagrangian L. However, unlike the Hamiltonian vector field in mechanics, Σ is not uniquely determined when $n > 1$.

This investigation studies the geometric structures that arise when the dynamical equations of the Carathéodory formulation are evaluated on nonsimple 2-vector fields, that is, sections of $\Lambda^2(M)$ that are not associated with plane fields on M. To introduce these equations, an n-vector field Σ on $\Lambda^n(M)$ is said to be regular if $\pi_*\Sigma$ is invertible.

Definition 1.1. An n-vector field $\Gamma \in \mathscr{E}_n(M)$ is said to satisfy the dynamical equations determined by a regular $\Sigma \in \mathscr{E}_n(\Lambda^n(M))$ with Legendre map $l = (\pi_* \Sigma)^{-1}$ if

$$\iota(l(\Gamma)_* \Gamma - \Sigma_{l(\Gamma)})\gamma = 0 \tag{1.3}$$

If Σ is the Hamiltonian n-vector field associated with a Lagrangian, then (1.3) evaluated on simple n-vector fields is the first-order or geodesic form of the Euler–Lagrange equations. For example, if the Lagrangian function is the squared length function of $\Lambda_n(M)$, then (1.3) implies the vanishing of the mean curvature tensor.

Fortunately, if Σ is a homogeneous n-vector field on $\Lambda^n(M)$, then (1.3) can be formulated as a system of equation on M. Let $X_\alpha \in \mathscr{X}(\Lambda^n(M))$ be the vector field that satisfies $\iota(X_\alpha)\gamma = \alpha$. The vector field X_α is a vertical vector field that serves as a homogeneity operator on functions along the fibers of $\Lambda^n(M)$.

Lemma 1.1. Let $\Sigma \in \mathscr{E}_n(\Lambda^n(M))$ be regular and satisfy the relation $L_{X_\alpha}\Sigma = c\Sigma$ and $d\iota(\Sigma)\gamma = 0$. An n-vector field Γ on M satisfies (1.3) if and only if

$$\iota(\Gamma)\, dl(\Gamma) - \frac{(-1)^n}{1+c}\, d(l(\Gamma)(\Gamma)) = 0 \tag{1.4}$$

Proof. See Martin (1988). ∎

Observe that the system (1.4) is completely determined by the Legendre map l. In most geometric applications of (1.4), $l\colon \Lambda_n(M) \to \Lambda^n(M)$ is the identification map induced by a metric on M; that is, if g is a metric on M, then g determines a map $l\colon TM \to T^*M$ given by $l(u)(v) = g(u, v)$, and l in turn induces a map denoted by the same symbol, $l\colon \Lambda_n(M) \to \Lambda^n(M)$. In the following all Legendre maps will be metric identification maps. Note that metric identification maps are homogeneous of degree one, and so $c = 1$. If l is induced by a metric and $n = 1$, then (1.4) is the defining system for geodesic vector fields on M. For other values of n, if Γ is an integrable simple vector field, then (1.4) gives the minimal surface equation for the leaves of Γ. However, (1.4) can equally well be applied to n-vector fields of any type. This article will study the system of equations generated when (1.4) is evaluated on nondegenerate 2-vector fields. It will be seen that this system is closely related to the dynamical equations of electrodynamics. In fact, if M is an Artinian manifold, (1.4) gives an extension of Maxwell's equations that is formally identical to the extension of Newtonian mechanics by relativistic mechanics.

2. THE MECHANICAL CASE

This section considers the relation between Newtonian mechanics and the mechanical structure of metric geometry from the point of view of equation (1.4). Although this relationship is elementary and well understood (Trautman, 1966), a reexamination of the questions involved is required to establish the connection between (1.4) and Maxwell's equations.

Recall that Newtonian mechanics can be formulated in terms of time-dependent vector fields on a Riemannian manifold S. A time-dependent vector field $V: S \times \mathbb{R} \to TS$ can be viewed as a vector field on $S \times \mathbb{R}$ by setting $\hat{V} = (V, 1)$. If $T = (0, 1)$, then the total time derivative of V is given by $\dot{V} = L_T \hat{V} + \nabla_V V$, where ∇ is the Levi-Civita connection on S. Given a real constant m and fixed time-dependent vector field F, a time-dependent vector field V satisfies Newton's second law with mass m and force F if $m\dot{V} = F$. Under special conditions it is possible to obtain Newton's second law as the geodesic equation of a metric geometry (M, g).

Proposition 2.1. Let T be a gradient Killing vector field on M with nonvanishing length. If V is a vector field on M such that $V = T + v$ and $g(T, v) = 0$, then $\nabla_V V = L_T v + \nabla_v v \equiv \dot{v}$.

Proof. If $T = \dot{\nabla} t$, then the constant-t hypersurfaces are totally geodesic. Consequently, since $\|T\|$ is constant, any vector field v that is orthogonal to T satisfies $\nabla_v T = 0$. Hence, $\nabla_{T+v} T + v = [T, v] + \nabla_v v$. ∎

Therefore a curve $\gamma(s)$ in M with $dt(\dot{\gamma}(s)) = 1$ is geodesic if and only if it satisfies Newton's second law. If M does not possess such a splitting, then Newton's second law and the geodesic equations determine different motions relative to any nondegenerate function. However, for any metric geometry and any constant-length vector field V, there is a locally defined nondegenerate function t such that relative to the splitting determined by t the total time derivative and the total covariant derivative agree at a point. To derive this fact, introduce the following general constructions on vector bundles.

Let $\pi: E \to N$ be a vector bundle over N, and let $\Gamma(E)$ be the space of smooth sections of E. Suppose that E possesses a linear connection ∇; that is, ∇ is a map $\nabla: \mathscr{X}(N) \times \Gamma((E) \to \Gamma(E)$ that over smooth functions on N is linear in the first entry and a derivation in the second entry. One can associate with ∇ the vertical map $V: TE \to VE$, where $VE \to E$ is the vertical subbundle of TE that is defined by the identity $\pi_* VE = 0$. If $X \in \mathscr{X}(N)$ and $s \in \Gamma(E)$, then V is related to ∇ by the identity $iV_s(s_* X) = \nabla_X s$. Here i: $VE \to E$ is the natural map that identifies a tangent to the fiber with a vector in the fiber. The vertical map iV and bundle map π determine a bundle isomorphism j: $TE \to E \oplus TN$ given by $j = iV \oplus \pi_*$. A tensor field T on E will be said to be a lifted tensor field for the connection ∇ if for any pair v_1, $v_2 \in E$ in the same fiber, $j(T(v_1)) = j(T(v_2))$. To each section of s of E there corresponds a lifted vertical vector field $\tilde{s}$ satisfying $i\tilde{s}(v) = s(\pi(v))$ and to each vector field X on N there is a lifted horizontal field $\tilde{X}$ satisfying $V\tilde{X} = 0$. It is not hard to see that if $X \in \mathscr{X}(N)$ and $s \in \Gamma(E)$, then the covariant derivative is also given by $\nabla_X s = i[\tilde{X}, \tilde{s}]$ (Dombrowski, 1962).

The vector bundles considered in this discussion are the normal bundles of submanifolds $N \hookrightarrow M$. If (M, g) is a metric geometry and g is nondegenerate on N, then the normal bundle of N may be identified with $TN^{\perp}$, the orthogonal complement of the tangent bundle to N. If U is a vector field defined along N, let $U^{\parallel}$ and $U^{\perp}$ denote the components of U in TN and $TN^{\perp}$. Recall that the Levi-Civita connection ∇ on M defines a connection $\nabla^{\perp}$ in $TN^{\perp}$ given by $\nabla^{\perp}_X V = (\nabla_X V)^{\perp}$ for $X \in \mathscr{X}(N)$ and $V \in \Gamma(TN^{\perp})$. Recall also that the shape tensor h is given by $\mathrm{h}(X, V) = (\nabla_X V)^{\parallel}$. Let $\exp_N : TN^{\perp} \to M$ be the exponential map of the metric geometry. If X is a vector field on $TN^{\perp}$, then $X^x = \exp_{N*} X$ is a vector field defined in a neighborhood of N. Also, if $\hat{\nabla}$ is a connection in $TN^{\perp}$, let S: $TN \times TN^{\perp} \to TN^{\perp}$ be the difference tensor $\hat{\nabla} - \nabla^{\perp}$.

Lemma 2.1. If $V \in \Gamma(TN^{\perp})$ and $X \in \mathscr{X}(N)$ and if $\tilde{X}$, $\tilde{V} \in \mathscr{X}(E)$ are the corresponding lifted vector fields for $\hat{\nabla}$, then

$$\nabla_{\tilde{V}^x} \tilde{X}^x |_N = \mathrm{h}(X, V) - S(X, V)$$

Proof. Since ∇ is symmetric,

$$\nabla_{\tilde{V}^x} \tilde{X}^x - \nabla_{\tilde{X}^x} \tilde{V}^x - [\tilde{V}^x, \tilde{X}^x] = 0$$

But $[\tilde{V}^x, \tilde{X}^x] = \exp_{N*}[\tilde{V}, \tilde{X}]$, and so $[\tilde{V}^x, \tilde{X}^x]|_N = [\tilde{V}, \tilde{X}] = -\hat{\nabla}_X V$. However,

$$\nabla_{\tilde{X}^x} \tilde{V}^x |_N = \nabla_X V$$

and so

$$\nabla_{\tilde{V}^x} \tilde{X}^x = \nabla_X V - \hat{\nabla}_x V = \mathrm{h}((X, V) - S(X, V) \quad \blacksquare$$

Using this structure, it is now possible to derive the reduction of the geodesic equations to Newton's second law in the Fermi coordinates surrounding a geodesic. A similar but more intricate argument will be used to prove an analogous result in electrodynamics.

Theorem 2.1. Let V be a vector field on M with constant length. For each $x \in M$ there is a neighborhood U of x and functions t and c on U such that $V = c\nabla t + v$ with $g(\nabla t, v) = 0$, $V(x) = \nabla t(x)$, and $\nabla_V V|_x = L_{\nabla t} v|_x = \dot{v}|_x$.

Proof. First observe that if $S \subset M$ is a nondegenerate curve, that is, g is nondegenerate on TS, and if p: $S \to \mathbb{R}$ is the arc length parametrization, then the function $\tau: TS^\perp \to \mathbb{R}$ defined by $\tau(v) = p(\pi(v))$ is a lifted function and $d\tau$ is a lifted 1-form. Let $t = \exp_S^{-1*}\tau$. Lemma 2.1 implies that if $U \in \Gamma(TS^\perp)$ and if X is a lifted field on $TS^\perp$, then

$$\nabla_{\bar{U}^x} dt(X^x)|_S = -dt(\mathrm{h}(\pi_* X, U))$$

Now suppose that V is a vector field on M with constant length and that S is an integral curve of V. Upon writing $V = c\nabla t + v$ for a function c chosen so that $V|_S = \nabla t|_S$ and $g(\nabla t, v) = 0$, one finds, using (1.4),

$$l(\nabla_V V)|_S = \iota(V)\, dl(V) + \tfrac{1}{2} dl(V)(V)|_S = (dc)^\perp + \iota(\nabla t)\, dl(V)|_S = (dc)^\perp|_S$$

Here $(dc)^\perp$ denotes the restriction of dc to $TS^\perp$. Since $\|V\|^2 = c^2\|\nabla t\|^2 + \|v\|^2$, $dc|_S = -\frac{1}{2} d\|\nabla t\|^2|_S$, and so for a lifted vertical field U,

$$dc(U^x)|_S = -\tfrac{1}{2} U^x \|\nabla t\|^2|_S = dt(\mathrm{h}(\nabla t, \mathrm{i}U))$$

Consequently, if S is a geodesic, then $dt(\mathrm{h}(\nabla t, \mathrm{i}U)) = 0$. Suppose that V' is a geodesic field such that $\|V'\|^2 = \|V\|^2$ and $V'(x) = V(x)$. If t' is the function determined by the geodesic S' through x, then V and V' can be decomposed by $\nabla t'$ as $V = c\nabla t' + v$ and $V' = c'\nabla t' + v'$. Now note that $(dc')^\perp|_x = -\frac{1}{2} d\|\nabla t'\|^{2\perp}|_x = (dc)^\perp|_x = 0$, and so the same calculation as above now gives

$$l(\nabla_V V)|_x = \iota(\nabla t')\, dl(v)|_x + (dc)^\perp|_x = L_{\nabla t'} l(v)|_x = l(L_{\nabla t'} v)|_x \quad \blacksquare$$

3. THE ELECTROMAGNETIC CASE

This section develops the connection between (1.4) evaluated no non-degenerate 2-vector fields and Maxwellian electrodynamics. The relevant Legendre map $l: \Lambda_2(M) \to \Lambda^2(M)$ is induced by an Artinian metric g on M. Recall that an Artinian manifold is a $2n$-dimensional manifold together with a metric of signature (n, n). A nondegenerate quadratic form is said to be of signature (p, q) if the form is positive on p basis elements and negative on q basis elements of an orthogonal basis. To demonstrate a general connection between (1.4) and Maxwell's equations, first consider the special case where (1.4) is precisely Maxwell's equations.

Example 3.1. Let (N, q) be a parallelizable Lorentzian manifold and let φ be a closed differential 2-form on N representing the electromagnetic field tensor. Let γ be the canonical symplectic form on T^*N. Recall that φ determines a translated symplectic form on T^*N given by $\omega = \gamma + \pi^*\varphi$. If $\jmath: T(T^*N) \to T^*(T^*N)$ is the identification map induced by γ, then $\Lambda = \jmath^{-1}\omega$ is a nondegenerate 2-vector field on T^*N. Two-vector fields of this type will be called electromagnetic 2-vector fields. Now let $(e_1, \ldots, e_n)$ be an orthonormal parallel frame field on N, and let $(e_1^*, \ldots, e_n^*)$ be the corresponding dual frame field; that is, $e_j^*(e_i) = \delta_{ji}$. If $(e_1, \ldots, e_n)$ is lifted horizontally to $(e_1, \ldots, e_n)$ and if $(e_1^*, \ldots, e_n^*)$ is lifted vertically to $(f_1, \ldots, f_n)$ one obtains a frame field $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ on T^*N. This frame field is a Darboux frame field for γ, and so $\gamma = \sum_i f_i^* \wedge e_i^*$. If $\varphi_{ij} = \varphi(e_i, e_j)$, then $\omega = \sum_i f_i^* \wedge e_i^* + \sum_{i<j} \varphi_{ij} e_i^* \wedge e_j^*$, and since $j(e_i) = -f_i^*$ and $j(f_i) = e_i^*$, then $\Lambda = \sum_i f_i \wedge e_i + \sum_{i<j} \varphi_{ij} f_i \wedge f_j$. Now suppose that an Artinian metric g is defined on T^*N by setting $g(e_i, e_j) = q(\pi_* e_i, \pi_* e_j)$, $g(f_i, f_j) = -q(e_i^*, e_j^*)$, and $g(f_i, e_j) = 0$. If l: $T(T^*N) \to T^*(T^*N)$ is the identification induced by g, then $l(\Lambda) = -\sum_i f_i^* \wedge e_i^* + \sum_{i<j} \pi_i \varphi_{ij} \varepsilon_j f_i^* \wedge f_j^*$, where $\varepsilon_i = q(e_i, e_j)$. Substituting these expressions into (1.4) gives

$$\iota(\Lambda)\, dl(\Lambda) - \tfrac{1}{2} d(l(\Lambda)(\Lambda)) = -4 \sum_{i,j} \varepsilon_i \nabla_{e_i} \varphi_{ij} \varepsilon_j f_j^* = 4 \sum_j (\operatorname{div} \varphi)_j f_j^*$$

This example shows that, as in mechanics, an extra condition must be placed on (1.4) in order for (1.4) to reduce to a linear model. The extra condition in this case is the flatness of the underlying space-time.

In the light of this calculation the question arises, is there a general relation between (1.4) evaluated on nondegenerate 2-vector fields and Maxwellian electrodynamics? The first step toward realizing this correspondence is to interpret (1.4) as the reduction of a second-order system. This is possible when (1.4) is evaluated on simple vector fields, since simple vector fields can be expressed in terms of derivatives of immersions. To find the correct notion of a potential for a nondegenerate 2-vector field, one appeals to electrodynamics, where one finds that the vector potential corresponds to a Lagrangian submanifold.

Given a nondegenerate 2-vector field Λ on a manifold M, a submanifold $N \hookrightarrow M$ is called a Lagrangian submanifold if the normal bundle to N, $\operatorname{ann}(TN) \subset T^*M|_N$, is a Lagrangian subbundle for Λ. On the cotangent bundle the vector potentials of an electromagnetic field are Lagrangian submanifolds of the 2-vector field introduced in Example 3.1.

Example 3.1 (*continued*). If A is a 1-form on N satisfying $dA = \varphi$, then the complement of $TA(N)$ relative to γ, $(TA(N))^\perp \subset T(T^*N)|_{A(N)}$, is a Lagrangian subbundle of $\omega = \gamma + \pi^*\varphi$. However, if $\lambda \in \operatorname{ann}(TA(N))$, then $j^{-1}(\lambda) \in TA(N)^\perp$, and so if $\Lambda = j^{-1}(\omega)$, then $A(N)$ is Lagrangian for Λ. Note that an electromagnetic 2-vector field is uniquely determined by any vector potential.

In general if the metric on T^*N is not flat, then (1.4) evaluated on electromagnetic 2-vector fields leads to an inconsistent system (Martin, 1988). However, if (1.4) is evaluated on a larger class of 2-vector fields on T^*N, then the vector potential no longer uniquely determines a corresponding 2-vector field, and so the interpretation of (1.4) as a second-order system is lost. This situation is comparable to the situation in relativistic mechanics where a parametrized curve no longer determines a unique velocity. However, (1.4), in its manifestation as the geodesic equations, determines a parametrization of solutions up to a scale, since the tangent to a solution must have constant length. A similar but not so obvious construction can be made when (1.4) is evaluated on nondegenerate 2-vector fields. In this case, the analog of the constant-length condition in mechanics appears to be the following condition.

Definition 3.1. A nondegenerate 2-vector field Λ is said to have charge $e \in \mathbb{R}$ if $\mathscr{C}(\Lambda, l(\Lambda)) = e\mathrm{id}$.

Example 3.2. There are many examples of 2-vector fields on an Artinian manifold satisfying Definition 3.1. If T^*N is the cotangent bundle of a metric geometry (N, q) and if $j\colon T(T^*N) \to T^*N \oplus TN$ is the bundle isomorphism induced by the Levi-Civita connection on N, then $g = j^*(-q \oplus q)$ is an Artinian metric on T^*N. It is not hard to see that if γ is the canonical 2-form, then $\mathscr{C}(l^{-1}(\gamma), \gamma) = -\mathrm{id}$. Other examples can be constructed from almost-Hermitian manifolds. For this construction see Example 3.3.

To develop the consequences of Definition 3.1, first note that a nondegenerate 2-vector field on a manifold M with metric g defines an isomorphism $h\colon T^*M \to TM$ given by $h(\lambda) = \iota(\lambda)\Lambda$. Together l and h define an automorphism field E on TM given by $E = hl$. This automorphism field has the following properties.

Lemma 3.1. If Λ is a charge-e, 2-vector field, then (1) E is skew-symmetric relative to g, (2) $E^2 = -e\mathrm{id}$, and (3) a pair of complementary distributions (X, Y) on M satisfying $EX = Y$ are orthogonal if and only if X is Lagrangian for $l(\Lambda)$.

Proof. Part 1 follows from the skew symmetry of Λ. To see part 2, let $p \in M$, and let $\lambda \in T^*M_p$ and $v \in TM_p$; then

$$\lambda \mathscr{C}(\Lambda, l(\Lambda))v = \Lambda(\lambda, l(\iota(l(v))\Lambda)) = -\Lambda(lhlv, \lambda) = -\lambda((hl)^2 v)$$

Therefore, $(hl)^2 = -e\mathrm{id}$. Finally, to see part 3 note that if X is Lagrangian for $l(\Lambda)$, then since $g(Ex, z) = l(\Lambda)(x, z)$, then $x \in X_p$ implies $Ex \in X_p^\perp$. Also, if $Ex \in X_p^\perp$, then $x \in X_p$. Consequently, $Y = X^\perp$. ∎

Note that if X is a Lagrangian distribution for Λ, then clearly $X^\perp$ is Lagrangian for $l(\Lambda)$, and so, as a consequence of part 3, $EX^\perp = X$ is also Lagrangian for $l(\Lambda)$; that is, Λ and $l(\Lambda)$ have the same Lagrangian distributions. Also note that parts 1 and 2 imply that the metric must satisfy $\|Ex\|^2 = e\|x\|^2$. Consequently, Definition 3.1 places a restriction on the possible signatures of g. If $\epsilon > 0$, then g is Riemannian and E defines an almost-Hermitian geometry. If $\epsilon < 0$, then g is Artinian and E determines an almost product structure on M. Because only the Artinian geometry admits Lorentzian submanifolds, it shall be assumed that $\epsilon < 0$. In fact, it is convenient to assume that $\epsilon = -1$. The cases where $\epsilon \neq -1$ can easily be obtained by rescaling Λ.

With these definitions it is now possible to conjecture that if (1.4) is evaluated on nondegenerate 2-vector fields, then an analogous result to Theorem 2.1 holds true. First, as indicated, the unit-length condition of Theorem 2.1 should be replaced by Definition 3.1. Next, since the potentials for a constant-charge 2-vector field have been identified with Lagrangian submanifolds, the natural generalization of Theorem 2.1 would be to show that along appropriately chosen Lagrangian submanifolds, (1.4) reduces at least infinitesimally to Example 3.1. To carry out this program, I will begin by studying the behavior of a charge-(-1), 2-vector field in the neighborhood of a Lagrangian submanifold.

Recall that if V is a real vector space and if V^* is the dual space, then $V \oplus V^*$ possesses a natural nondegenerate 2-form γ_0 given by $\gamma_0((v, \lambda), (u, \mu)) = \lambda(u) - \mu(v)$ for (v, λ), $(u, \mu) \in V \oplus V^*$. If ω is a nondegenerate 2-form on a real vector space W and if $X, Y \subset W$ are complementary Lagrangian subspaces, then ω induces an isomorphism $i_0 \colon Y \to X^*$ given by $i_0(u) = \imath(u)\omega$ for $u \in X$. It is not hard to see that $\mathrm{id} \oplus i_0 \colon W \to X \oplus X^*$ is a symplectic isomorphism.

Now let Λ be a charge-(-1) 2-vector field and let $N \hookrightarrow M$ be a Lagrangian submanifold for Λ on which g is nondegenerate. As noted above, $(TN, TN^{\perp})$ defines a pair of complementary distributions for $l(\Lambda)$ along N. The map $i \colon TN^{\perp} \to (TN)^* \simeq T^*N$ defined by $i(v) = \iota(v) l(\Lambda)$ for $v \in TN^{\perp}$ is a bundle isomorphism between $TN^{\perp}$ and T^*N. If 0_N^* denotes the zero section of T^*N, then

$$T(T^*N)|_{0_N^*} = TN \oplus (TN)^*$$

and if γ is the canonical 2-form on T^*N, then

$$\gamma|_{0_N^*} = \gamma_0$$

If 0_N denotes the zero section of $TN^{\perp}$, then $T(TN^{\perp})|_{0_N} = TN \oplus TN^{\perp}$ and $i_*|_{0_N} = \mathrm{id} \oplus i_0$, and so

$$l(\Lambda)|_{0_N} = i^* \gamma|_{0_N^*}$$

Next observe that if $j \colon T(T^*N) \to TN \oplus T^*N$ is the isomorphism induced by the Levi-Civita connection $\nabla^{\parallel}$ on N, then γ is a lifted 2-form. To see this, first note that if $p \in T^*N$, then for $v \in V(TN)_p$ and $w \in T(T^*N)_p$, $\gamma(v, w) = i(v)(\pi_* w)$. Therefore, if for $(z_0, z_1) \in TN \oplus T^*N$, $j^{-1}(z_0, z_1)$ is written as

$j^{-1}(z_0, z_1) = z^H + i^{-1}z_1$, where z^H is the horizontal vector satisfying $\pi_* z^H = z_0$, then since the horizontal distribution is Lagrangian,

$$\gamma(j^{-1}(z_0, z_1), j^{-1}(y_0, y_1)) = z_1(y_0) - z_0(y_1)$$

Consequently, γ is lifted relative to the Levi-Civita connection. Now the bundle isomorphism i: $TN^\perp \to T^*N$ induces a metric connection $\hat{\nabla} = i^{-1}\nabla^\| i$ on $TN^\perp$, and so relative to the connection $\hat{\nabla}$, $i^*\gamma$ is a closed lifted 2-form on $TN^\perp$. If

$$\omega_0 = \exp_N^{-1*} i^*\gamma$$

then ω_0 is a closed nondegenerate 2-form defined in a neighborhood of N such that $\omega_0|_N = l(\Lambda)|_N$. The 2-form ω_0 will be used in much the same way as the gradient of the time function was used in Theorem 2.1.

The behavior of ω_0 near N depends on the connection $\hat{\nabla}$. To express $\hat{\nabla}$ in terms of $\nabla^\perp$, note that i: $TN^\perp \to T^*N$ can be written in terms of l and h as $i(w) = \iota(w)\, l(\Lambda) = lhl(w)$. Since $\nabla l = 0$, it follows from the fact that $E^2 = 1$ that for $X \in \Gamma(TN^\perp)$ and any $v \in TN$,

$$i^{-1}\nabla^\| i_v X - \nabla^\perp{}_v X = (E(\nabla_v E)(X))^\perp \tag{3.1}$$

From a similar computation, a metric connection $\tilde{\nabla}$ on M can be obtained with the property that $\tilde{\nabla}\Lambda = 0$. The connection $\tilde{\nabla}$ is constructed as follows.

Lemma 3.2. If $\tilde{\nabla} = \nabla + \frac{1}{2}E(\nabla E)$, then $\tilde{\nabla}$ is a metric connection with torsion and $\tilde{\nabla}\Lambda = 0$.

Proof. Introduce the connection $\nabla' = i^{-1}\nabla i$ on M that extends the connection $\hat{\nabla}$ in $TN^\perp$. Since $i = lhl$ is conformal, it follows that ∇' is a metric connection on M. Next observe that the dual connection ∇'^* in T^*M is just $\nabla'^* = i\nabla i^{-1}$, and consequently, for X, $Y \in \mathscr{X}(M)$,

$$(\nabla'_X i)(Y) = \nabla'^*_X i(Y) - i(\nabla'_X Y) = i(\nabla_X Y) - \nabla_X i(Y) = -(\nabla_X i)(Y)$$

If $S(X, Y) = \nabla'_X Y - \nabla_X Y$ and if for a 1-form λ, $S^T(X, \lambda)(Y) = \lambda(S(X, Y))$, then it follows from the above identity that

$$2(\nabla_X i)(Y) = S^T(X, iY) + iS(X, Y)$$

Consequently, if $\tilde{\nabla} = \nabla + \frac{1}{2}S$, then $(\tilde{\nabla}_X i)(Y) = 0$. As in (3.1), $S = E(\nabla E)$, and so the lemma now follows. ∎

Using the geometric structures introduced above, it is possible to derive a second-order system of PDEs for the Lagrangian submanifolds of a charge-(-1) 2-vector field Λ satisfying (1.4). To introduce this system of equations, recall that if $(e_1, \ldots, e_n)$ is an orthonormal frame field on N, then the mean curvature vector H of N is given by $H = \sum_i (\nabla_{e_i} e_i)^\perp$. Since N is a Lagrangian and $ETN = TN^\perp$, there is a second orthogonally invariant vector field along N, namely $\hat{H} = \sum_i (\nabla_{e_i} Ee_i)^{\|}$. Also introduce the following traces of the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$. For $V \in \Gamma(TN^{\perp})$, let $\operatorname{tr} \tilde{T}^{\perp}(V) = \sum_i l(\Lambda)(\tilde{T}(V, Er_i), e_i)$, and for $X \in \mathscr{X}(N)$, let

$$\operatorname{tr} \tilde{T}^{\|}(X) = \sum_i l(\Lambda)(T(X, e_i), Ee_i)$$

Theorem 3.1. If $N \hookrightarrow M$ is a Lagrangian submanifold for a charge-(-1) 2-vector field Λ satisfying (1.4), then

$$\operatorname{tr} \tilde{T}^{\|} = 0 \tag{3.2}$$

and

$$\tfrac{1}{2}(H + E\hat{H}) = 2l(\operatorname{tr} \tilde{T}^{\perp}) \tag{3.3}$$

Before proving this theorem, first consider what interpretation might be given the trace of the torsion tensor.

Example 3.3. Let ∇ be the Levi-Civita connection determined by the metric g on T^*N given in Example 3.2, and let D be the Levi-Civita connection on N with curvature R. Suppose that $\tilde{X}$ and $\tilde{Y}$ are horizontal lifts of vector fields X and Y on N, and suppose that $\tilde{U}$ and $\tilde{V}$ are vertical lifts of the 1-forms $l(U)$ and $l(V)$, where U and V are vector fields on N. It is well known that for $v \in T^*N$,

$$\nabla_{\tilde{V}} \tilde{X}_v = \tfrac{1}{2}\tilde{R}(v, V)X$$

$$\nabla_{\tilde{X}} \tilde{V} = i^{-1}(D_X lV) + \tfrac{1}{2}\tilde{R}(v, V)X$$

$$\nabla_{\tilde{X}} \tilde{Y} = \tilde{D}_X Y + i^{-1} lR(X, Y)v$$

$$\nabla_{\tilde{U}} \tilde{V} = 0$$

Now it is easy to see that for a lifted vertical field V, $EV = \widetilde{l(iV)}$. The definition of $\tilde{\nabla}$ implies $\tilde{T}(W, Z) = \tfrac{1}{2}[E(\nabla_W E)(Z) - E(\nabla_Z E)(W)]$ for $W, Z \in \mathscr{X}(T^*N)$, and so $\tilde{T}(\tilde{X}, \tilde{Y})_v = \tfrac{1}{4} i^{-1} lR(Y, X)v$ and $\tilde{T}(\tilde{U}, \tilde{U})_v = \tfrac{1}{4} i^{-1} lR(U, V)v$. Taking traces, one finds $\operatorname{tr} \tilde{T}^{\|} = 0$ and $\operatorname{tr} \tilde{T}^{\perp}(\tilde{V})_v = \tfrac{1}{4}\operatorname{Ric}(V, v)$, where Ric is the Ricci curvature of D.

Proof of Theorem 3.1. Equations (3.2) and (3.3) are derived by expressing (1.4) relative to the local symplectic ω_0 defined above. Let $\Lambda_0 = l^{-1}(\omega_0)$. By definition $\Lambda_0|_N = \Lambda|_N$, and if $\Sigma = \Lambda - c\Lambda_0$, where $c = \omega_0(\Lambda)/\omega_0(\Lambda_0)$, then $\omega_0(\Sigma) = 0$, $c|_N = 1$, and $\Sigma|_N = 0$. If $\sigma = l(\Sigma)$, then substituting $\Lambda = c\Lambda_0 + \Sigma$ and $l(\Lambda) = c\omega_0 + \sigma$ into (1.4) and using (1.2) gives

$$\iota(\Lambda_0)\, d\sigma - 2dc(\mathscr{C}(\Lambda_0, \omega_0)) + \omega_0(\Lambda_0)\, dc|_N = 0 \tag{3.4}$$

Now since $\mathscr{C}(\Lambda, l(\Lambda))$ is constant, it follows that $c^2\omega_0(\Lambda_0) + \sigma(\Sigma)$ is constant, and so on N, $2\omega_0(\Lambda_0)\, dc = -(4n)\, dc = -d\omega_0(\Lambda_0)$. Clearly, if $X \in \mathscr{X}(N)$, then

$d\omega_0(\Lambda_0)(X) = 0$. To find the vertical derivative of $\omega_0(\Lambda_0)$, let $U, V \in \Gamma(TN^\perp)$. Lemma 2.1, (3.1), the fact that ω_0 is the image of a lifted 2-form, and the fact that N is a Lagrangian imply that

$$\nabla_V \omega_0(U, X)|_N = -\omega_0(h(X, V), U)|_N \tag{3.5}$$

where $\mathrm{h}(X, V) = (\nabla_X V)^\parallel$ is the shape tensor of N. Let $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ be an orthonormal Darboux frame for $l(\Lambda)$ along N. Extend this frame to a neighborhood of N by lifting to $TN^\perp$ via $\hat{\nabla}$ and exponentiating. Denote the extended frame also by $(e_1, \ldots, e_n, f_1, \ldots, f_n)$. Now, $\Lambda_0|_N = l^{-1}(\omega_0)|_N = -\Sigma_i f_i \wedge e_i$. It follows from (3.5) that if $V \in \Gamma(TN^\perp)$, then

$$Vl^{-1}(\omega_0)(\omega_0)|_N = 2l^{-1}(\omega_0)(\nabla_V \omega_0)|_N = -4 \sum_i \omega_0(\mathrm{h}(e_i, V), f_i)|_N$$

and so

$$dc(V)|_N = -\frac{1}{n} \sum_i \omega_0(\mathrm{h}(e_i, V), f_i)|_N$$

The next step in evaluating (3.4) is to compute $\iota(\Lambda_0)\, d\sigma|_N$. First consider the orthogonal component $\iota(\Lambda_0)\, d\sigma^\perp|_N$. For any $W \in \Gamma(TM|_N)$,

$$\iota(\Lambda_0)\, d\sigma(W)|_N = -2\left[\sum_i \nabla_{f_i}\sigma(e_i, \tilde{W}^x) + \nabla_{\tilde{W}^x}\sigma(f_i, e_i) + \nabla_{e_i}\sigma(\tilde{W}^x, f_i)\right]\Bigg|_N \tag{3.6}$$

Since $\sigma = l(\Lambda) - c\omega_0$, clearly for $V \in \Gamma(TN^\perp)$, $\sum_i \nabla_{e_i}\sigma(V, f_i)|_N = 0$, and a calculation shows that

$$\sum_i \nabla_{f_i} c\omega_0(e_i, V)|_N = \sum_i -\frac{1}{n}\, \omega_0(\mathrm{h}(e_i, V), f_i) - \omega_0(\mathrm{h}(e_i, f_i), V)|_N$$

$$\sum_i \nabla_V c\omega_0(f_i, e_i)|_N = 0$$

These computations isolate all the terms in (3.4) that involve the shape tensor. Noting that (3.4) reduces to $\iota(\Lambda_0)\, d\sigma - 2(n+1)\, dc|_N = 0$ and substituting the above expressions, we find that (3.4) becomes

$$\left\{\sum_i \omega_0(\mathrm{h}(e_i, V), f_i) + \omega_0(\mathrm{h}(e_i, f_i), V)\right\}$$
$$-\left\{\sum_i \nabla_V l(\Lambda)(f_i, e_i) + \nabla_{f_i} l(\Lambda)(e_i, V)\right\}\Bigg|_N = 0$$

To compute the terms involving $\nabla l(\Lambda)$, note that for any $X, Y, Z \in \mathscr{X}(M)$

$$\nabla_X l(\Lambda)(Y, Z) - \nabla_Y l(\Lambda)(X, Z) = 2l(\Lambda)(\tilde{T}(X, Y), Z) \tag{3.7}$$

Substituting (3.7) into the above expression then leads to (3.3). To obtain (3.2), evaluate $\iota(\Lambda_0)\, d\sigma^{\|}|_N$ on a vector field $X \in \mathscr{X}(N)$. It is clear from (3.6) that $\iota(\Lambda_0)\, d\sigma(X)|_N = \sum_i \nabla_{f_i}\sigma(e_i, X)$. However, for $V \in \Gamma(TN^{\perp})$ and $X, Y \in \mathscr{X}(N)$, $\nabla_V l(\Lambda)(X, Y) = 0$. To see this, first note that for $p \in N$ the second fundamental form of $S = \exp_p TN^{\perp}$ vanishes at p and so $(\nabla_V EU)^{\|} = 0$ for all $V, U \in \Gamma(TN^{\perp})$. Consequently for all $V, U, W \in \Gamma(TN^{\perp})$,

$$\nabla_V l(\Lambda)(U, W) = g((\nabla_V E)(U), W) = g((\nabla_V EU)^{\|}, W) = 0$$

Since $l(\Lambda)(U, V) = -l(\Lambda)(EU, EV)$ it now follows that $\nabla_V l(\Lambda)(X, Y) = 0$. As a result,

$$\iota(\Lambda_0)\, d\sigma(X)|_N = \sum_i \nabla_{f_i}\omega_0(e_i, X) \tag{3.8}$$

Since ω_0 is closed, (3.7) gives

$$\sum_i \nabla_{f_i}\omega_0(e_i, X) = \sum_i \nabla_{e_i}\omega_0(f_i, X) + \nabla_X\omega_0(e_i, f_i) = 2\omega_0(\tilde{T}(X, e_i), f_i) \quad \blacksquare$$

The proof of Theorem 3.1 is also essentially the argument that demonstrates that along Lagrangian submanifolds (1.4) is infinitesimally equivalent to Maxwell's equations. This result is analogous to Theorem 2.1; however, unlike mechanics, where (1.4) reduces to the force-free Newton second law, the approximate Maxwell equations may possess a current. To obtain this result, let Λ be a charge-(-1) 2-vector field with a nondegenerate Lagrangian submanifold N. Suppose that Λ' is a second charge-(-1) 2-vector field with a Lagrangian submanifold N' that solves (1.4) such that at $p \in M$, $\Lambda_p = \Lambda'_p$ and $TN_p = TN'_p$. Let $\tilde{T}$ and $\tilde{T}'$ be the torsions determined by Λ and Λ', respectively.

Theorem 3.2. If ω'_0 is the 2-form defined by Λ' in a neighborhood of N' and if $l(\Lambda) = c\omega'_0 + \sigma$ with $c(p) = 1$ and $\sigma(p) = 0$, then

$$\iota(\Lambda)\, dl(\Lambda) + \tfrac{1}{2}\, dl(\Lambda)(\Lambda)|_p = -2\left[\sum_i (\iota(f_i)\nabla_{e_i}\sigma)^{\perp} + 2(\operatorname{tr} \tilde{T}^{\perp} - \operatorname{tr} \tilde{T}'^{\perp})|_p\right]$$

Proof. Decompose Λ relative to $\Lambda'_0 = l^{-1}(\omega'_0)$ by $\Lambda = c\Lambda'_0 + \Sigma$, so that $c(p) = 1$ and $\Sigma(p) = 0$. Since Λ has charge-(-1), $dc_p = (1/4n)\, d\omega'_0(\Lambda'_0)$. Note that the terms of $\iota(\Lambda_0)\, d\sigma^{\perp}$ that are determined by the shape tensor depend only on ω'_0 and p. Consequently, from (3.3),

$$\iota(\Lambda)\, dl(\Lambda)^{\perp}|_p = \sum_i (\iota(f_i)\nabla_{e_i}\sigma)^{\perp} + (\operatorname{tr} \tilde{T}^{\perp} - \operatorname{tr} \tilde{T}'^{\perp})|_p$$

However, from (3.8) the parallel component of $\iota(\Lambda)\, dl(\Lambda)$ depends only on ω'_{0p} and so by (3.2) it must vanish at p. $\blacksquare$

Clearly, if $l(\Lambda)$ is closed, then (1.4) is trivially satisfied and so (3.2) and (3.3) hold along all Lagrangian submanifolds of $l(\Lambda)$. As a consequence it is easy to see from Example 3.2 that for any Lagrangian submanifold of a charge-(-1) 2-vector field, there is a 2-vector field Λ' satisfying the hypotheses of Theorem 3.2. Therefore Theorem 3.2 has the following corollary.

Corollary 3.1. If $N \hookrightarrow M$ is a Lagrangian submanifold of a charge-(-1) 2-vector field Λ, then the 1-form $\iota(\Lambda)\, dl(\Lambda)|_N \in \operatorname{ann}(TN)$.

If, however, $l(\Lambda)$ is not closed, then the existence of Lagrangian submanifolds is not automatic. In fact, if $\dim(M) > 8$, then generically a nondegenerate 2-form ω possesses no Lagrangian submanifolds. This follows from the fact that if $N \hookrightarrow M$ is a Lagrangian submanifold for ω, then TN must annihilate both ω and $d\omega$. However, if $\dim(M) > 8$ and if ω and $d\omega$ are both generic, there are no $[\frac{1}{2}\dim(M)]$-dimensional distributions that annihilate both ω and $d\omega$. The following example shows that there do exist nontrivial solutions to (3.2) and (3.3) if $\dim(M) \leq 8$.

Example 3.4. Examples of Artinian manifolds with charge-(-1) 2-vector fields where $l(\Lambda)$ is not closed can be constructed from almost-Hermitian geometries with a real orthogonal splitting. Let M be an almost-Hermitian manifold with almost complex structure J and Hermitian metric q, and suppose $TM = X \oplus Y$, where X and Y are real orthogonal distributions M with $JX = Y$. Define a new metric g on M by setting $g = -q|_X \oplus q|_Y$ and define an almost product structure E on M by setting $E|_X = -J|_X$ and $E|_Y = J|_Y$. The automorphism E is now skew-symmetric relative to g and the Kähler form is given by $\omega(u, v) = g(Eu, v)$ for $u, v \in TM_p$. If l_q and l_g are the identifications determined by q and g, respectively, then $l_g^{-1}(\omega)$ has charge -1, while $l_q^{-1}(\omega) = -l_g^{-1}(\omega)$ has charge $+1$. Now from Proposition 4.2, (1.4) is equivalent to $\operatorname{div}_\omega l_g^{-1}(\omega) = -\operatorname{div}_\omega l_q^{-1}(\omega) = 0$. Almost-Hermitian manifolds that satisfy the latter equality are called semi-Kählerian and they are called balanced if the almost complex structure is integrable. Balanced manifolds enjoy many interesting properties (Michelsohn, 1983). A simple example of a balanced manifold that is not Kähler is the three-dimensional solvable complex Lie group,

$$G = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \middle| (a, b, c) \in \mathbb{C}^3 \right\}$$

The complex structure is induced by left translation of multiplication by i on the Lie algebra of G,

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{bmatrix} \middle| (a, b, c) \in \mathbb{C}^3 \right\}$$

The Hermitian metric and Kähler form ω are left translates of the standard Hermitian metric and Kähler form on $\mathbb{C}^3$. However, G is not a Kähler manifold, since ω is not closed. In fact, if (e_1, e_2, e_3) is the standard basis in $\mathbb{C}^3$, then $d\omega(ae_1, be_2, ce_3)=\frac{1}{2}(\overline{ab}c - ab\bar{c})$. Since $\Lambda|_I = l^{-1}(\omega)|_I = -(e_1 \wedge ie_1 + e_2 \wedge ie_2 + e_3 \wedge ie_3)$, it follows that $\iota(\Lambda)\, d\omega = 0$ and so Λ is a solution to (1.4). Clearly, TG possesses a real Lagrangian splitting and so G has a compatibile Artinian structure. In fact, ω possesses a Lagrangian submanifold on which the Artinian metric is Lorentzian. Note that $\mathscr{h} = \{(ia, ib, c)|(a, b, c)\in\mathbb{R}^3\}$ is a Lagrangian subalgebra of $\mathscr{g}$, and so

$$\exp \mathscr{h} = \left\{ \begin{bmatrix} 1 & ia & c-ab \\ 0 & 1 & ib \\ 0 & 0 & 1 \end{bmatrix} \middle| (a, b, c)\in\mathbb{R}^3 \right\}$$

is a Lagrangian submanifold for ω. Identifying e_3 with the time direction, one sees that in this case the approximating solution of Maxwell's equation is the electrostatic solution with the potential $\phi(a, b) = -ab$.

4. CONSERVATION LAWS

This section presents several consequences of (1.4) that correspond to the current and energy-momentum differential conservation laws in Maxwellian electrodynamics. These identities will further substantiate the relation between (1.4) and electrodynamics. To introduce these relations, recall the definition of the Schouten bracket on 2-vector fields (Nijhenhuis, 1955; Martin, 1988). The Schouten bracket is a differential pairing of 2-vector fields that extends the Lie derivative on vector fields. On simple 2-vector fields it is given by

$$[U\wedge V, X\wedge Y] = X\wedge(L_Y U\wedge V) + (L_X U\wedge V)\wedge Y$$

The utility of the Schouten bracket is that it greatly simplifies the Palais formula for the exterior derivative. If ω is a 3-form and if M and N are 2-vector fields, then the Palais formula has the form

$$d\omega(M, N) = \iota(M)\, d\iota(N)\omega + \iota(N)\, d\iota(M)\omega + \omega([M, N]) \tag{4.1}$$

Using (4.1), we can derive differential conservation laws for 2-vector fields. To state these laws, let j be the 1-form determined by (1.4), that is,

$$j = \iota(\Lambda)\, dl(\Lambda) - \tfrac{1}{2} dl(\Lambda)(\Lambda) \tag{4.2}$$

Also, define the energy-momentum tensor associated with Λ by

$$\mathscr{E}(\Lambda, l(\Lambda)) = -\mathscr{C}(\Lambda, l(\Lambda)) + \tfrac{1}{4}\operatorname{tr}\mathscr{C}(\Lambda, l(\Lambda))\mathrm{id} \tag{4.3}$$

The quantities j and $\mathscr{E}$ satisfy the following identities.

Theorem 4.1. Let Λ be a 2-vector, and let f be a smooth function on M; then Λ and f satisfy

$$\iota(\Lambda)\, dj = -\tfrac{1}{2}\, dl(\Lambda)([\Lambda, \Lambda]) \tag{4.4}$$

$$df(\iota(j)\Lambda) = \iota(\Lambda)\, d(df\mathscr{C}(\Lambda, l(\Lambda))) + \tfrac{1}{4} df \wedge l(\Lambda)([\Lambda, \Lambda]) \tag{4.5}$$

Proof. Equation (4.4) is a direct consequence of (4.1) and (4.2). To obtain (4.4), substitute $\omega = df \wedge l(\Lambda)$ and $M = N = \Lambda$ in (4.1). Next introduce a frame field $(e_1, \ldots, e_{2n})$ such that $\Lambda = \sum_{i=1}^{n} e_i \wedge e_{i+n}$. The first step is to rewrite the left-hand side of (4.1). From the definitions given in Section 1,

$$\begin{aligned} df \wedge dl(\Lambda)(\Lambda, \Lambda) &= 4 \sum_{i,j=1} df \wedge l(\Lambda)(e_i, e_{i+n}, e_j, e_{j+n}) \\ &= 8 \sum_{i,j=1} df(e_i)\, dl(\Lambda)(e_{i+n}, e_j, e_{j+n}) \\ &\quad - df(e_{i+n})\, dl(\Lambda)(e_i, e_j, e_{j+n}) \end{aligned}$$

However,

$$\iota(\Lambda)\iota(\Lambda)\, dl(\Lambda) = 2 \sum_{i,j=1}^{n} dl(\Lambda)(e_j, e_{j+n}, e_i) e_{i+n} - dl(\Lambda)(e_j, e_{j+n}, e_{i+n}) e_i$$

and so $df \wedge dl(\Lambda)(\Lambda, \Lambda) = -4\, df\, \iota(\Lambda)\iota(\Lambda)\, dl(\Lambda)$. Next observe that (1.2) implies that $\iota(\Lambda)\, df \wedge l(\Lambda) = -2\, df(\mathscr{C}(\Lambda, l(\Lambda))) + l(\Lambda)(\Lambda)\, df$. Substituting these two relations into (4.2) gives

$$\begin{aligned} 4\, df\, \iota(\Lambda)\iota(\Lambda)\, dl(\Lambda) &= 2\iota(\Lambda)\, d(-2df(\mathscr{C}(\Lambda, l(\Lambda))) + l(\Lambda)(\Lambda)\, df) \\ &\quad + df \wedge l(\Lambda)([\Lambda, \Lambda]) \end{aligned} \tag{4.6}$$

Now note that

$$2df(\imath(\Lambda)\, dl(\Lambda)(\Lambda)) = 2\Lambda(dl(\Lambda)(\Lambda), df) = \imath(\Lambda)\, d(l(\Lambda)(\Lambda)\, df)$$

Subtracting this identity from (4.6) gives (4.5). ∎

A direct calculation shows that in Example 3.1, (4.4) is just the current conservation law of classical electrodynamics. This follows since for an electromagnetic 2-vector field Λ on T^*N, $[\Lambda, \Lambda] = 0$. On the other hand, if Λ is a constant-charge 2-vector field, then both (4.2) and (4.4) can be written divergences relative to the volume element $\Omega = \bigwedge_{i=1}^{n} l(\Lambda) = l(\Lambda)^n$. If ω is a nondegenerate 2-form and Σ is a k-vector field, then let $\operatorname{div}_\omega \Sigma$ be the $(k-1)$-form defined by the relation $\operatorname{div}_\omega \Sigma \wedge \omega^{n-k+1} = d\iota(\Sigma)\omega^n$.

Proposition 4.1. If Λ is a constant-charge 2-vector field, then $j = \operatorname{div}_{l(\Lambda)} \Lambda$ and $\iota(\Lambda)\, dj = \operatorname{div}_{l(\Lambda)} \iota(j)\Lambda = 0$.

Proof. The proof is similar to the proof of a corresponding set of identities in Hermitian geometry. It relies on the relation that if $\mathscr{C}(\Lambda, \omega) = 1$, then for any k-form α, $(\iota(\Lambda)\alpha) \wedge \omega^{n-k+1} = k(n-k+1)\alpha \wedge \omega^{n-k}$; for details see Goldberg (1970), pp. 168–182. ∎

To interpret (4.5), first note that if $[\Lambda, \Lambda]=0$, then (4.5) is formally identical to the conservation law of energy-momentum in electrodynamics. To appreciate this correspondence, consider this identity in the context of Example 3.1.

Example 3.1 (*continued*). Using the notation previously introduced, a computation shows that for an electromagnetic 2-vector field $\Lambda=\sum_i f_i \wedge e_i+\sum_{i,j}\varphi_{ij} f_i \wedge f_j$ and a dual 2-form $l(\Lambda)=-\sum_i f_i^* \wedge e_i^*+\sum_{i,j}\varepsilon_i\varphi_{ij}\varepsilon_j f_i^* \wedge f_j^*$, the energy-momentum tensor defined by (4.3) has the form

$$\mathscr{E}(\Lambda, l(\Lambda))=\sum_{i,j}\left(2\varphi_{ij} f_i \otimes e_j^*-2\varepsilon_i\varphi_{ij}\varepsilon_j e_i \otimes f_j^*-4\sum_k \varphi_{ik}\varepsilon_k\varphi_{kj}\varepsilon_j f_i \otimes f_j^*\right)$$
$$+\tfrac{1}{4}\left[4\sum_{i,j}\varphi_{ij}\varepsilon_i\varphi_{ij}\varepsilon_j-(2n-4)\right]\sum_i f_i \otimes f_i^*+e_i \otimes e_i^*$$

To compute the right-hand side of (4.5), choose df so that $df=f_i^*$ and set $\mathscr{E}_i(\Lambda, l(\Lambda))=f_i^*(\mathscr{E}(\Lambda, l(\Lambda)))$. A calculation shows that

$$\iota(\Lambda)\, d\mathscr{E}_i(\Lambda, l(\Lambda))=8\sum_{j,k}\nabla_{e_k}\left(\varphi_{ji}\varepsilon_j\varphi_{jk}\varepsilon_k-\tfrac{1}{4}\delta_{ik}\left(\sum_{m,l}\varphi_{ml}\varepsilon_m\varphi_{ml}\varepsilon_l\right)\right)$$

which is the ith component of the divergence of the energy-momentum tensor associated with the field strength φ. Next let $df=e_i^*$ and $\mathscr{E}_i(\Lambda, l(\Lambda))=e_i^*(\mathscr{E}(\Lambda, l(\Lambda)))$. In this case,

$$\iota(\Lambda)\, d\mathscr{E}_i(\Lambda, l(\Lambda))=4\sum_k \nabla_{e_k}\varepsilon_i\varphi_{ik}\varepsilon_k$$

which is the ith component of the divergence of φ. If $j=\sum_i j_i f_i^*$ is the current defined by (4.2), then $\iota(j)\Lambda=\sum_i j_i e_i-2(\sum_k \varphi_{ik} j_k) f_i$, and so (4.5) expresses both the field equations and the conservation law of energy-momentum. In some sense $\mathscr{E}$ can be compared with the invariant 4-momentum in special relativity, since its divergence contains both the energy-momentum conservation law and the field equations.

If Λ is a constant-charge 2-vector field, then clearly $\mathscr{E}(\Lambda, l(\Lambda))$ is a constant multiple of the identity and in this case (4.5) becomes

$$\iota(j)\Lambda=\tfrac{1}{2}\iota(l(\Lambda))[\Lambda, \Lambda] \tag{4.7}$$

This relation is just an alternative form of (4.2) and could also have been derived from the relation $dl(\Lambda)=\frac{1}{2}i[\Lambda, \Lambda]$. Recall that $i=lhl$ and here i is applied to 3-vectors. Consequently, when (4.5) is evaluated on a constant-charge 2-vector field one obtains no new relations. This fact again indicates that the constant-charge 2-vector model extends classical electrodynamics in much the same manner in which relativistic mechanics extends Newtonian mechanics.

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